ON NONLINEAR DIFFRACTION OF WEAK SHOCK WAVES

PMM Vol. 35, №3, 1971, pp. 492-498
L. I. MOGILEVICH and G. P. SHINDIAPIN
(Saratov)
(Received December 10, 1970)

The nonlinear problem of diffraction of a weak shock wave near a rigid wall with a kink is examined. The solution of the problem in linear formulation leads to the appearance of a singularity at the point of intersection of the initial and diffracted fronts. Subsequent approximations do not remove this singularity. The reported attempt by de Mestre [1] to remove this singularity leads to considerable mathematical difficulties which cannot be completely overcome.

In this paper the method of matched asymptotic expansions [2] is applied, and a uniformly applicable solution of the problem is constructed in the first approximation. The solution of the problem in the acoustic formulation [3, 4] is used for the external solution. The internal problem is reduced to the solution of the nonlinear problem in the theory of short waves [5].

1. Let us examine the propagation of a plane weak shock wave through a quiescent, ideal, polytropic gas along a rigid wall with a kink α. The system of coordinates is selected as shown in Fig. 1a. The rate of propagation of the shock wave U_0 is determinated by its intensity

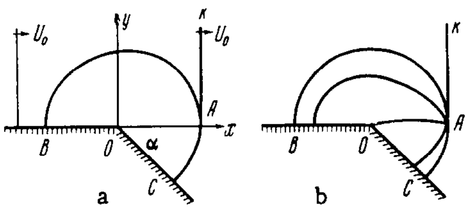

Fig. 1

$$\varepsilon = \frac{p_1 - p_0}{\varkappa p_0} = \frac{2}{\varkappa + 1}\left[\left(\frac{U_0}{a_0}\right)^2 - 1\right], \qquad \varepsilon \ll 1 \tag{1.1}$$

Here p, a, $\varkappa$ are pressure, speed of sound and the ratio of heat capacities, respectively. The subscript 0 corresponds to gas parameters ahead of the shock wave, the subscript 1 behind the shock wave. At the instant $t = 0$ the shock wave passes through the point of the kink in the wall 0. As the shock wave propagates further along the inclined wall OC, expansion waves arise. Through interaction with these expansion waves the shock front AC is bent while the pressure along the front declines from the value p_1 at the point A to the value p_0 in the direction of the wall. The diffraction (perturbation) region is closed from above by the boundary AB which represents the front of the wave of perturbation arising at the point 0 and the time $t = 0$.

In the diffraction region $CABC$ the components of velocity u, v, the pressure p, the density ρ, and the entropy S satisfy the equations of dynamics (motion, continuity, state, energy)

$$u_t + uu_x + vu_y + \frac{1}{\rho}p_x = 0, \qquad v_t + uv_x + vv_y + \frac{1}{\rho}p_y = 0 \tag{1.2}$$

$$\rho_t + (u\rho)_x + (v\rho)_y = 0, \qquad p = p(\rho, S), \qquad S_t + uS_x + vS_y = 0$$

For weak shock waves we can neglect the entropy change in the entire region of diffraction and to assume the flow to be irrotational to the order of ε^2 inclusive. Let us introduce the velocity potential Φ

$$\Phi_x = u, \qquad \Phi_y = v$$

and write the equation of state in the following form:

$$\varkappa P = \frac{p - p_0}{p_0} = \left[\left(\frac{\rho}{\rho_0}\right)^{\varkappa} - 1\right] \tag{1.3}$$

The absence of characteristic dimensions of length and time in the problem permits to introduce dimensionless independent and dependent variables

$$x = a_0 tr\cos\theta, \quad y = a_0 tr\sin\theta, \quad \Phi = a_0^2 tf(r, \theta) \tag{1.4}$$
$$u = a_0 u^*, \quad v = a_0 v^*, \quad a = a_0 a^*, \quad p = \rho_0 a_0^2 p^*, \quad \rho = \rho_0\rho^*, \; U_0 = a_0 U_0^*$$

From equations of motion (1.2) and (1.3), utilizing (1.4), we can obtain an equation [1, 6] for the potential f

$$\begin{aligned}(1 - r^2) f_{rr} + \frac{1}{r} f_r + \frac{1}{r^2} f_{\theta\theta} &= (\varkappa - 1)(f - rf_r)\left(f_{rr} + \frac{1}{r} f_r + \frac{1}{r^2} f_{\theta\theta}\right) + \\ &+ \frac{2}{r^2} f_\theta (f_\theta - rf_{r\theta}) - 2rf_r f_{rr} + \frac{1}{r^4} f_\theta^2 f_{\theta\theta} + \frac{2}{r^2} f_r f_\theta f_{r\theta} + f_r^2 f_{rr} - \\ &- \frac{1}{r^3} f_r f_\theta^2 + \frac{\varkappa - 1}{2}\left(f_r^2 + \frac{1}{r^2} f_\theta^2\right)\left(f_{rr} + \frac{1}{r} f_r + \frac{1}{r^2} f_{\theta\theta}\right)\end{aligned} \tag{1.5}$$

and also the Bernoulli equation

$$a^2 = (1 + \varkappa P)^{\varkappa - 1/\varkappa} = 1 - (\varkappa - 1)\left(f - rf_r + \frac{1}{2} f_r^2 + \frac{1}{2r^2} f_\theta^2\right) \tag{1.6}$$

The superscripts on variables have been omitted. Let us formulate the conditions on the boundary of the diffraction region. On the rigid wall BOC

$$f_\theta = 0 \text{ for } \vartheta = -\alpha \text{ and } \vartheta = \pi \tag{1.7}$$

The location of points $r = r(\vartheta)$ of the front of perturbation AB is determined through the velocity of sound a_1 and the normal component of the gas velocity with respect to the front u_{1n} by the equation

$$r\left[1 + \left(\frac{r'}{r}\right)^2\right]^{-1/2} = u_{1n} + a_1, \quad u_{1n} = \left(f_{1r} - \frac{r'}{r^2} f_{1\theta}\right)\left[1 + \left(\frac{r'}{r}\right)^2\right]^{-1/2} \tag{1.8}$$

On the perturbation front AB the following conditions of continuity of velocity and pressure hold

$$f_r = f_{1r}, \qquad f_\theta = f_{1\theta}, \qquad P = P_1 \tag{1.9}$$

Here the velocity potential f_1 and the relative excess pressure P_1 of the homogeneous flow behind the front of the shock wave AK have the following form according to (1.1)

$$f_1 = \varepsilon(cr\cos\theta - 1), \qquad c = \left(1 + \frac{\varkappa + 1}{2}\varepsilon\right)^{-1/2}, \qquad P_1 = \varepsilon \tag{1.10}$$

The equation of the shock wave front AC has the form

$$r = k(\theta) \tag{1.11}$$

On the shock wave front AC the following conditions of dynamic compatibility hold

$$u_n = \frac{2}{\varkappa + 1}[U_0 - U_0^{-1}], \qquad U_0 = k\left[1 + \left(\frac{k'}{k}\right)^2\right]^{-1/2} \tag{1.12}$$
$$u_\tau = 0, \qquad P = \frac{p - p_0}{\varkappa p_0} = \frac{2}{\varkappa + 1}[U_0^2 - 1]$$

Expressing the normal component u_n and the tangential component u_τ of the velocity behind the shock wave front in terms of the components of velocity f_r and f_θ, we obtain the following differential equation of the front from conditions (1.12):

$$\frac{2}{\varkappa+1}\left\{k\left[1+\left(\frac{k'}{k}\right)^2\right]^{-1}-k^{-1}\right\}=f_r \tag{1.13}$$

We also obtain the conditions

$$k'f_r+f_\theta=0, \qquad P=kf_r \tag{1.14}$$

In this manner the problem is reduced to the integration of a system of nonlinear equations (1.5) and (1.6) with boundary conditions (1.7)–(1.9), (1.13) and (1.14). This is connected with considerable mathematical difficulties.

2. In the construction of a solution for the diffraction region of weak shock waves it is customary to use the method [1, 6] of asymptotic expansion with respect to the small parameter ε

$$f(r,\theta,\varepsilon)=\varepsilon f^{(1)}(r,\theta)+\varepsilon^2 f^{(2)}(r,\theta)+\ldots, \qquad P=\varepsilon P^{(1)}+\varepsilon^2 P^{(2)}+\ldots \tag{2.1}$$

For the first terms of expansion of $f^{(1)}$ and $P^{(1)}$ we obtain from equations (1.5), (1.6) a system of linear equations

$$(1-r^2)f_{rr}^{(1)}+\frac{1}{r}f_r^{(1)}+\frac{1}{r^2}f_{\theta\theta}^{(1)}=0, \qquad P^{(1)}=rf_r^{(1)}-f^{(1)} \tag{2.2}$$

In the linear formulation the front of the shock wave

$$r=1+\varepsilon k^{(1)}(\theta)+\varepsilon^2 k^{(2)}(\theta)+\ldots \tag{2.3}$$

is examined as a front of weak perturbations ($f_r=f_\theta=0$) and we obtain in accordance with (1.13)

$$f_r^{(1)}=\frac{4}{\varkappa+1}k^{(1)}=0, \qquad k^{(1)}(\theta)\equiv 0 \tag{2.4}$$

The boundary conditions (1.7)–(1.9), (1.13) and (1.14) for potential $f^{(1)}$ in the first approximation assume the following form

$$\begin{aligned} &f_\theta^{(1)}=0 \quad \text{for} \quad 0\leqslant r\leqslant 1,\ \vartheta=-\alpha, \text{ and } \vartheta=\pi \\ &f_r^{(1)}=\cos\theta, \quad f_\theta^{(1)}=-\sin\vartheta \quad \text{for} \quad r=1,\ 0<\theta\leqslant\pi \\ &f_r^{(1)}=0, \quad f_\theta^{(1)}=0, \quad \text{for} \quad r=1, \quad -\alpha\leqslant\theta<0 \end{aligned} \tag{2.5}$$

Utilizing the Chaplygin transformation

$$\sigma=r^{-1}(1-\sqrt{1-r^2}) \tag{2.6}$$

and eliminating $f^{(1)}$ from the system (2.2), we obtain the Laplace equation for $P^{(1)}$

$$P_{\sigma\sigma}^{(1)}+\frac{1}{\sigma}P_\sigma^{(1)}+\frac{1}{\sigma^2}P_{\theta\theta}^{(1)}=0 \tag{2.7}$$

with boundary conditions according to (2.2), (2.3), (2.5)

$$\begin{aligned} &P_\theta^{(1)}=0 \quad \text{for} \quad 0\leqslant\sigma\leqslant 1,\ \theta=-\alpha \text{ and } \theta=\pi \\ &P^{(1)}=1 \quad \text{for} \quad \sigma=1,\ 0<\theta\leqslant\pi \\ &P^{(1)}=0 \quad \text{for} \quad \sigma=1,\ -\alpha\leqslant\theta<0 \end{aligned} \tag{2.8}$$

The solution of Eq. (2.7) with boundary conditions (2.8) was examined in a series of papers [3, 4] and can be represented in the form [1, 4]

$$P^{(1)}=1-\frac{1}{\pi}\operatorname{arctg}\left\{\frac{1-\sigma^\lambda}{1+\sigma^\lambda}\operatorname{ctg}\frac{\lambda}{2}\theta\right\}+ \tag{2.9}$$

$$+\frac{1}{\pi}\operatorname{arctg}\left\{\frac{1-\sigma^{\lambda}}{1+\sigma^{\lambda}}\operatorname{ctg}\frac{\lambda}{2}(\theta+2\alpha)\right\}\qquad \lambda=\frac{\pi}{\pi-\alpha}$$

In the case of negative values of the argument, the values of arctangents in (2.9) are taken on the other branch according to the formula

$$\operatorname{arctg}\eta=\pi-\operatorname{arctg}(-\eta)\tag{2.10}$$

In Fig. 1b the pressure field in the diffraction region is shown qualitatively according to the solution (2.9). At the point A $(r=1,\ \theta=0)$ the solution (2.9) has a singularity. The pressure $P^{(1)}$ changes in a jump from the value $P^{(1)}=0$ on AC to $P^{(1)}=1$ on AB. This singularity is a result of the physical defect in the acoustic formulation when the pressure along the diffraction front AC is constant while the front itself represents an acoustic region $r=1$. The formulation of the second approximation does not remove the indicated singularity, as was shown in [1]. This leads to the necessity to examine the problem of diffraction as a problem of singular perturbations [2] for the expansion (2.1).

3. In order to construct a solution which removes the singularity of the acoustic theory in the vicinity of the point A, we change to internal variables [2]

$$r=1+{}^1/_2(\varkappa+1)\varepsilon^{\gamma}\delta,\qquad \theta=\sqrt{{}^1/_2(\varkappa-1)}\,\varepsilon^{\omega}Y\tag{3.1}$$

The potential f and the pressure P are represented in the form

$$f=\varepsilon^{\alpha}\frac{\varkappa+1}{2}F^{(1)}(\delta,Y)+\ldots,\qquad P=\varepsilon^{\beta}P^{(1)}+\ldots\tag{3.2}$$

Substituting (3.1) and (3.2) into the system (1.5), (1.6) and conditions (1.8), (1.9), (1.13), (1.14) and comparing the orders of higher terms of the left and right sides in Eqs. (1.5), (1.6), we obtain

$$\alpha=2,\ \beta=1,\ \gamma=1,\ \omega={}^1/_2\tag{3.3}$$

Introducing the notation

$$F_{\delta}^{(1)}=\mu,\qquad F_{Y}^{(1)}=\nu$$

we obtain according to (1.5) and (1.6) the following system of nonlinear equations for the first terms of expansion (3.2) (single-term internal expansion):

$$2(\mu-\delta)\mu_{\delta}+\nu_{Y}+\mu=0,\quad \mu_{Y}=\nu_{\delta},\quad P^{(1)}=\mu\tag{3.4}$$

Conditions (1.8) and (1.9) on the boundary of the diffraction region assume the form (Fig. 2a)

$$\mu=1,\ \nu=-Y,\ P^{(1)}=1\quad\text{for }\delta=1\tag{3.5}$$

From (1.13) and (1.14) we obtain the equation of the shock wave front $A'C'$

$$d\delta/dY=\sqrt{2\delta-\mu}\tag{3.6}$$

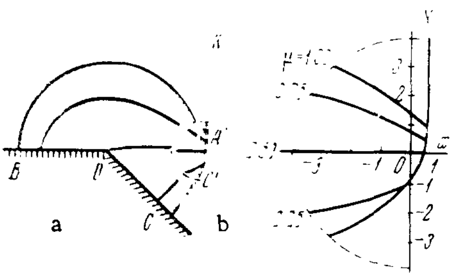

Fig. 2

and the following conditions on the front

$$\mu=P^{(1)},\qquad \mu\,d\delta/dY+\nu=0\tag{3.7}$$

The Hugoniot condition on the front $\mu=P^{(1)}$ is satisfied automatically, according to the third equation of (3.4). Making use of (3.5)–(3.7), we obtain in the point A' where the front $A'C'$ intersects with the line $A'B$

$$\mu=1,\ \nu=-1,\quad\text{for }\delta=1,\ Y=1\tag{3.8}$$

We note that for values of $\alpha \gg \varepsilon^{1/2}$ the condition on the wall BOC

$$v = 0 \quad \text{for} \quad Y = -\frac{\alpha}{\sqrt{1/_2(\varkappa + 1)\,\varepsilon}} \quad \text{and} \quad Y = \frac{\pi}{\sqrt{1/_2(\varkappa + 1)\,\varepsilon}} \tag{3.9}$$

must be discarded as external.

The system of equations (3.4) represents a known system of equations for short waves [5, 7]. We shall seek a solution for Eq. (3.4) with boundary conditions (3.5)–(3.7) in the following form [5]:

$$\delta = -{}^1/_2 Y^2 \operatorname{tg}^2(b\mu + c) + B\sin^2(b\mu + c) + {}^1/_2 b^{-1}\sin 2(b\mu + c) + \mu \tag{3.10}$$

$$v = [b^{-1}\operatorname{tg}(b\mu + c) - \mu]\, Y, \qquad b,\, c,\, B = \text{const}$$

Satisfying condition (3.5), we obtain $c = -b$. The arbitrary constant b is determined in the process of matching the external expansion (2.9) with the internal expansion (3.10). The constant B is determined in the process of satisfying the condition of conservation of the tangential component (3.7) on the front $A'C'$.

4. We follow the method of matched asymptotic expansions [2], we express the external solution (2.9) in internal variables (3.1) and retain the first term in the expansion with respect to ε (single-term internal expansion). Then, taking into account (2.10), we obtain

$$P^{(1)} = 1 - \frac{1}{\pi}\operatorname{arctg}\frac{\sqrt{-2\delta}}{Y} \tag{4.1}$$

Writing the internal solution (3.10) in external variables r and θ (3.1) and retaining the first term of the expansion with respect to ε (single-term external expansion), we obtain the following expression by rewriting the result in internal variables δ and Y:

$$\mu = P^{(1)} = 1 + \frac{1}{b}\operatorname{arctg}\frac{\sqrt{-2\delta}}{Y} \tag{4.2}$$

Comparing (4.1) and (4.2), we determine the value of the constant

$$b = -\pi \tag{4.3}$$

5. The differential equation of the shock wave front according to (3.6), (3.10), (4.3) has the form

$$\frac{d\mu}{dY} = \frac{YL^2M + M\sqrt{(2BL^2 - 2\pi^{-1}LM + \mu)M^4 - Y^2L^2M^2}}{\pi Y^2L + M^3(M^2 - L^2 + 1 - 2\pi BLM)} \tag{5.1}$$

$$L = \sin\pi(1 - \mu), \qquad M = \cos\pi(1 - \mu)$$
$$B = \text{const}$$

We integrate Eq. (5.1) and construct the shock wave front from the point A', where $\mu = 1$ and $Y = 1$. Then, satisfying the condition of conservation of tangential component (3.7) at the point with the coordinate $Y = 0$ where according to (3.10) $\mu = {}^1/_2$, we obtain the value $B = 0.45$ for the constant B.

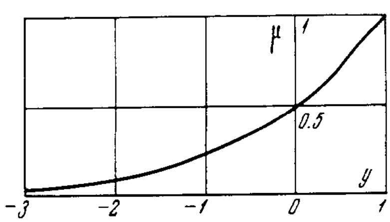

Fig. 3

In Fig. 2a the qualitative picture of pressure distribution is shown in the diffraction region. Constant pressure (velocity) lines near the front $A'C'$ are constructed in Fig. 2b according to solution (3.10) in the system of coordinates $Y = Y$ and $X = \delta - {}^1/_2 Y^2$ [7]. The pressure distribution near (Fig. 2b) and along (Fig. 3) the shock wave front $A'C'$ is consistent with the pressure distribution in the external region where the flow described by the solution (2.9) and for the condition $\alpha \gg \varepsilon^{1/2}$, does not depend on the angle of the kink of the wall. For angles $\alpha \sim 0\ (\varepsilon^{1/2})$ where the condition $\alpha \gg \varepsilon^{1/2}$ is not satisfied, solutions (2.9) and (3.10) cannot be used. In this case the pressure along the front $A'C'$ falls to the value $p_c \geqslant p_0$ on the wall and the boundary condition (3.9) on the wall $v = 0$ for $Y = -\alpha / \sqrt{{}^1/_2 (\varkappa + 1) \varepsilon}$ must be already satisfied in the solution of the internal problem (3.4)–(3.7).

In conclusion we note that a solution of the form (3.10) was utilized in [8] for the construction of flow near the front of the reflected wave in the problem of shock wave reflection from a rigid wall with a kink. With reference to the remarks presented above it appears to the authors, however, that it is not justified to use solutions (3.10) in the problem on reflection [8] in the case of small angles of the kink of the wall.

The authors express their gratitude to S. V. Fal'kovich for advice in the discussion of this work.

BIBLIOGRAPHY

1. De Mestre, N. J., Nonlinear weak shock diffraction. J. Austral. Math. Soc. Vol. 8, Pt. 4, pp. 737-754, 1968.
2. Van Dyke, M. D., Perturbation Methods in Fluid Mechanics. New York, Academic Press, 1964.
3. Sobolev, S. L., Theory of plane wave diffraction. Tr. Seismol. Inst. №41, 1934.
4. Sagomonian, A. A., Spatial Problems of Unsteady Motion of a Compressible Fluid. M., Izd. Mosk. Univ. 1962.
5. Ryzhov, O. S. and Khristianovich, S. A., On nonlinear reflection of weak shock waves. PMM Vol. 22, №5, 1958.
6. Lighthill, M. J., The shock strength in supersonic "conical fields". Philos. Mag. Vol. 40, №311, 1949.
7. Shindiapin, G. P., Regular reflection of weak shock waves from a rigid wall. PMM Vol. 29, №1, 1965.
8. Bagdoev, A. G. and Gurgenian, A. A., Two nonlinear problems on definition of regions of shock waves in a fluid. Izv. Akad. Nauk ArmSSR, "Mekhanika", №4, 1968.

Translated by B. D.